Bend-a-ball: Make Flexible Zonohedra and Other Polyhedra with a Simple Modular Origami Edge Unit

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Abstract

In this workshop we will make modular origami structures that bend and collapse, commonly known as flexiballs. However, we will use an improved and simplified unit that makes for better results. Within the 90 minutes of the workshop, we should be able to make the 12-unit cube/rhombohedron and/or the 24-unit rhombic dodecahedron: these are degree-3 and degree-4 polar zonohedra (PZ). 20 units make a partial icosahedron. Experienced folders may wish to begin to make the 40-unit degree-5 PZ rhombic icosahedron or 60-unit rhombic triacontahedron. Practical considerations are the number of units and desired colours. One classic colouring scheme is to use the same colour for parallel edges, another is the same colour for edges meeting at a vertex.

Introduction

Polyhedra are natural subjects for modular origami. The Platonic solids are popular, as are some Archimedean solids [5]. Typically, the origami units act as either *face*, *vertex* or *edge* units. To make a given polyhedron, we might choose face units as this usually minimises the number of units needed. Choosing edge units usually maximises the number of units needed, so what are the benefits of this apparently more onerous approach? One benefit is the versatility of edge units: the same unit can make different angles by assembling different numbers of units at a vertex. Another benefit is the edge units can be used to make *wireframe origami*, typically polyhedral frames that interpenetrate each other, e.g. five intersecting tetrahedra and other polypolyhedra [7, 6].

However, some edge units can be awkward to assemble at a vertex: it can be like trying to close a door to a room from the outside – but only using the door handle inside the room. The flexible nature of zonohedra means that the vertex can be isolated and worked on more easily, e.g. see the right hand tip of Figure 1).

Furthermore, the vertices of edge units can flex leading to a form of action modular origami [3, 4].

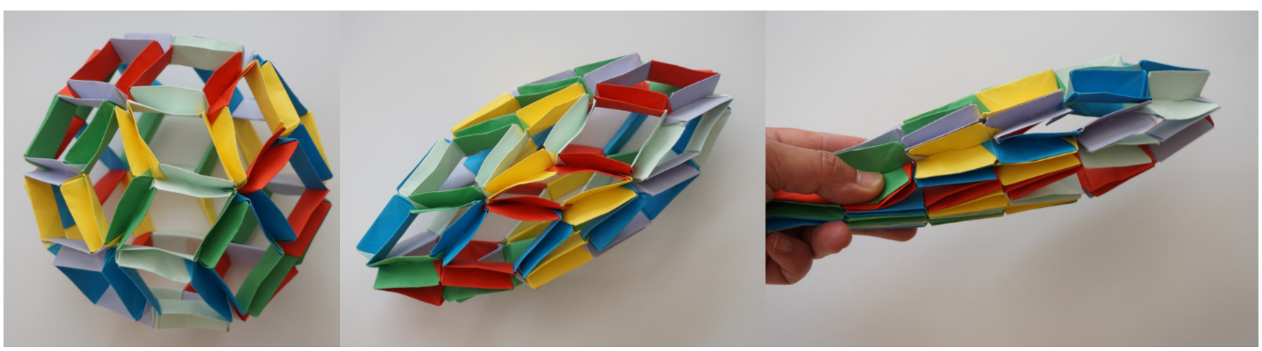

Figure 1: *Rhombic Triacontahedron collapsing*

Number of Units and Colours

First, choose the number of units that you can make in the given time. Within the workshop, you should be able to make at least 12 units and have sufficient time to assemble the cube (Figure 2). You may have enough time to try the 24-unit rhombic dodecahedron or the 20-unit partial icosahedron (Figures 8 and 9).

A simple colouring scheme is to make all units the same colour. A little more sophisticated is to use the same colour for edges meeting at some vertices. However, I feel it is most instructive to make parallel edges the same colour (and sometimes the most challenging). Here are the numbers needed for the three most appealing constructions:

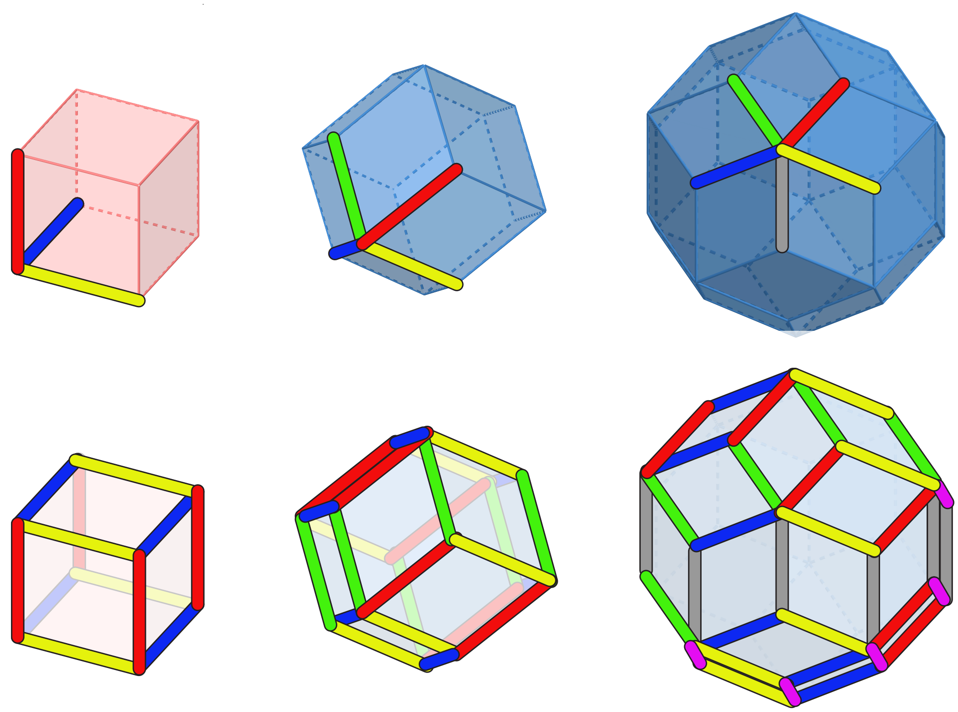

Figure 2: *Using the same colour for parallel edges.*
Left: cube. Middle: rhombic dodecahedron. Right: rhombic triacontahedron.
Top row: each edge of a star is a different colour.
Bottom row: opposite edges of each rhombus are the same colour.

Cube

Make 12 units in three colours. All vertices have three edges meeting. For convenience, use three sheets in different colours. Before cutting each sheet into quarters, divide the short edge into eighths: this precreasing makes the first few module steps easier.

Rhombic Dodecahedron

Make 24 units in four colours. Vertices have either three or four edges meeting.

Rhombic Triacontahedron

Make 60 units in six colours. Vertices have either three or five edges meeting. Think of the starting vertex as the north pole: the sixth colour appears at the equator and the south pole has the colours of the north pole in reverse order.

Zonohedra

The cube and rhombic dodecahedron are 3-fold and 4-fold polar zonohedra (PZ), respectively. However, the rhombic triacontahedron is a zonohedron but not a PZ: a 5-fold PZ is a rhombic icosahedron (Figure 3). For a rhombic icosahedron, 5 faces meet at the poles, 3 meet at the vertices next to the poles and 4 faces meet at all other vertices. A zone is the set of faces sharing one edge direction, i.e. the edges are parallel. Half of a a zone is highlighted below left. Table 1 shows the number of faces, edges and vertices of selected PZ. The number of distinct face shapes is, at most, $n/2$ rounded down to a whole number [2].

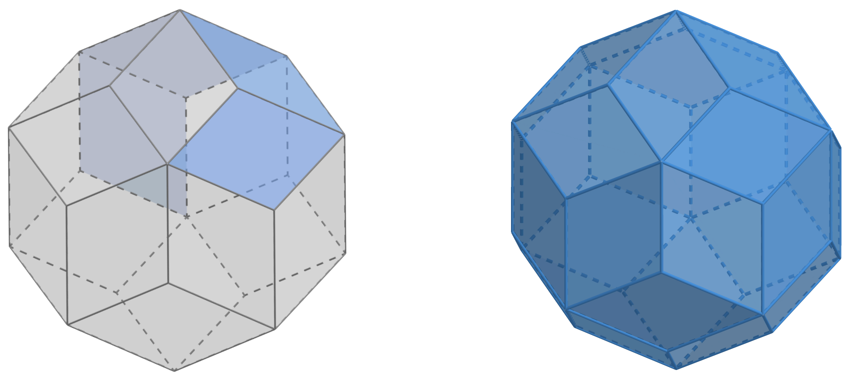

Figure 3: *Left: rhombic icosahedron, a polar zonohedron of degree 5. Right: rhombic triacontrahedron, a zonohedron but not a polar zonohedron.*

Table 1: *Number of faces, edges, vertices and distinct faces for polar zononhedra of degree 3 to 20*

Degree	Faces	Edges	Vertices	Distinct Faces	Edges per Degree
n	F	E	V	F_d	E_n
	$n(n-1)$	$2n(n-1)$	$n(n-1)+2$	$floor(n/2)$	E/n
3	6	12	8	1	4
4	12	24	14	2	6
5	20	40	22	2	8
6	30	60	32	3	10
7	42	84	44	3	12
8	56	112	58	4	14
9	72	144	74	4	16
10	90	180	92	5	18
11	110	220	112	5	20
12	132	264	134	6	22
13	156	312	158	6	24
14	182	364	184	7	26
15	210	420	212	7	28
16	240	480	242	8	30
17	272	544	274	8	32
18	306	612	308	9	34
19	342	684	344	9	36
20	380	760	382	10	38

Materials and Optional Cutting Plan

Two approaches are possible:

- Fold one sheet of paper for each unit. Use memo cube paper as a convenient source of colourful squares: the edge length is usually around 10 cm.
- Alternatively, prepare larger sheets and cut into smaller sheets for folding.

When starting with larger sheets like A4 or letter size paper, the simplest and most accurate method for cutting is to divide each sheet into either four, eight or 16 rectangles per sheet. However, six or twelve rectangles are almost as convenient and may be needed when you only have a limited amount of colour paper (Figure 4). Although scissors indicate which lines to cut, more accurate results come from folding along the line and slitting with a knife or certain kinds of paper cutters (typically made for opening envelopes).

You can precrease the folds used in step 1: before cutting four, divide the short edge into eighths. For six, divide the long edge into twelfths. For twelve, divide the short edge into twelfths. Use valley folds in all divisions.

You can also prepare the crease in step 5: however, the accuracy may be worse than folding each unit separately.

Note that machine-made paper has grain, which is usually parallel to the longer edge of the oblong. Try to align the first valley folds along the grain for stronger units. The effect is more noticeable for longer oblongs, say > 1.5:1. However, longer units can be easier to assemble as the two ends interfere less with each other. They also make some assemblies more effective, e.g. the partial icosahedron (Figure 8).

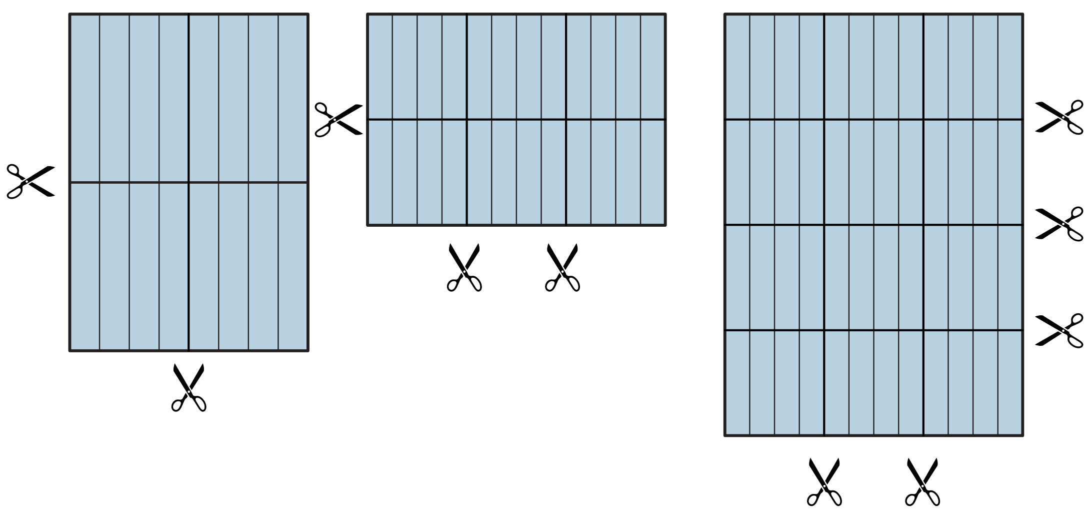

Figure 4: *Cutting four, six and twelve rectangles from an oblong, e.g. A4 or letter size paper*

The Module

Figure 5 shows how to fold a unit and Figure 6 shows how to assemble three units at a vertex. The unit arose whilst trying to create a different 30-piece assembly: I had found a joining mechanism to make structures like Jorge Pardo's *Flex-a-ball* [9, 8] using a simpler join, yet still as effective. The units can be reused to make other shapes since the units are easy to disassemble. The joint is related to that used by Tomoko Fuse in her *Action Lizard* [1, p. 34].

Folding the unit

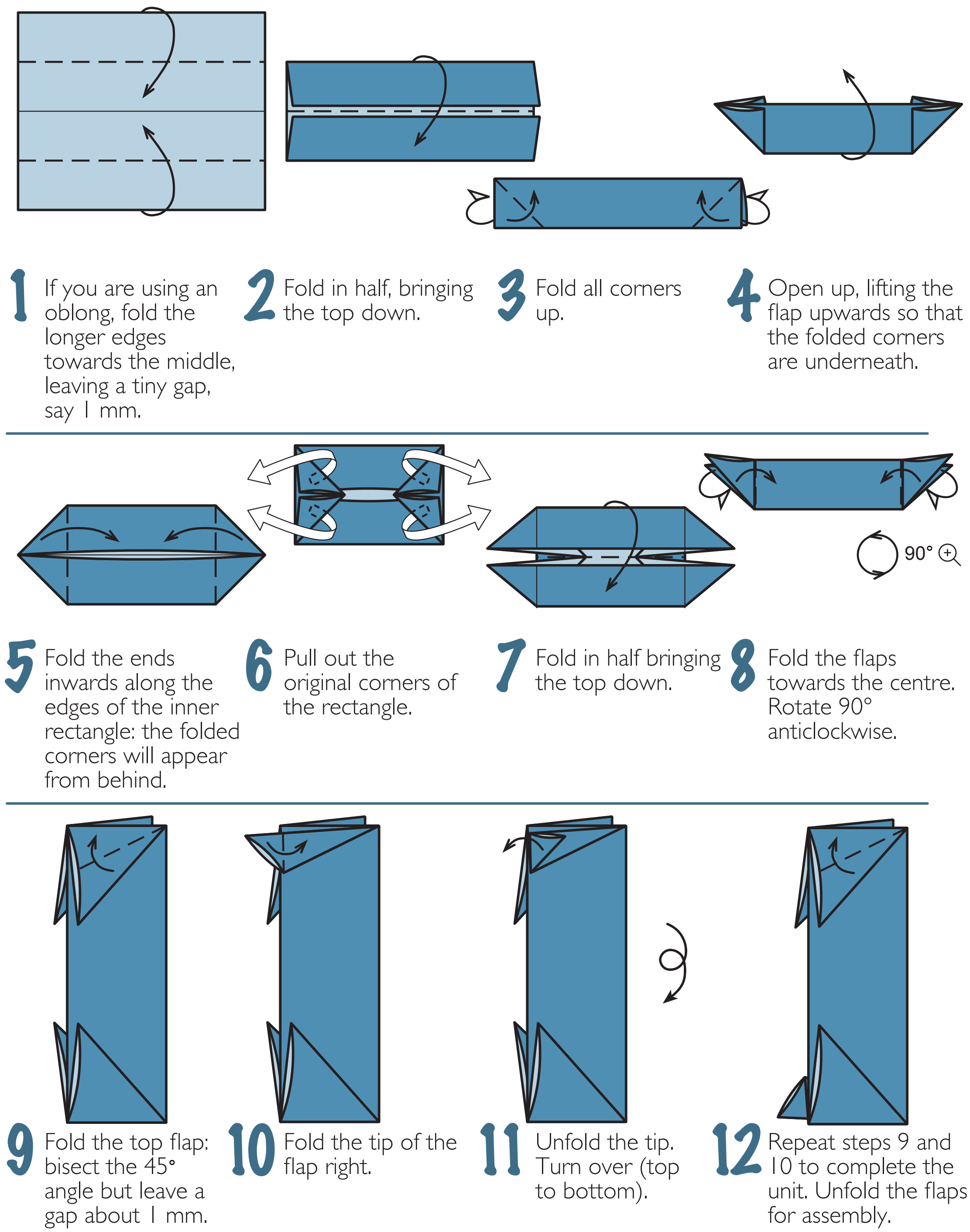

Figure 5: *Folding the unit*

Assembling a Vertex

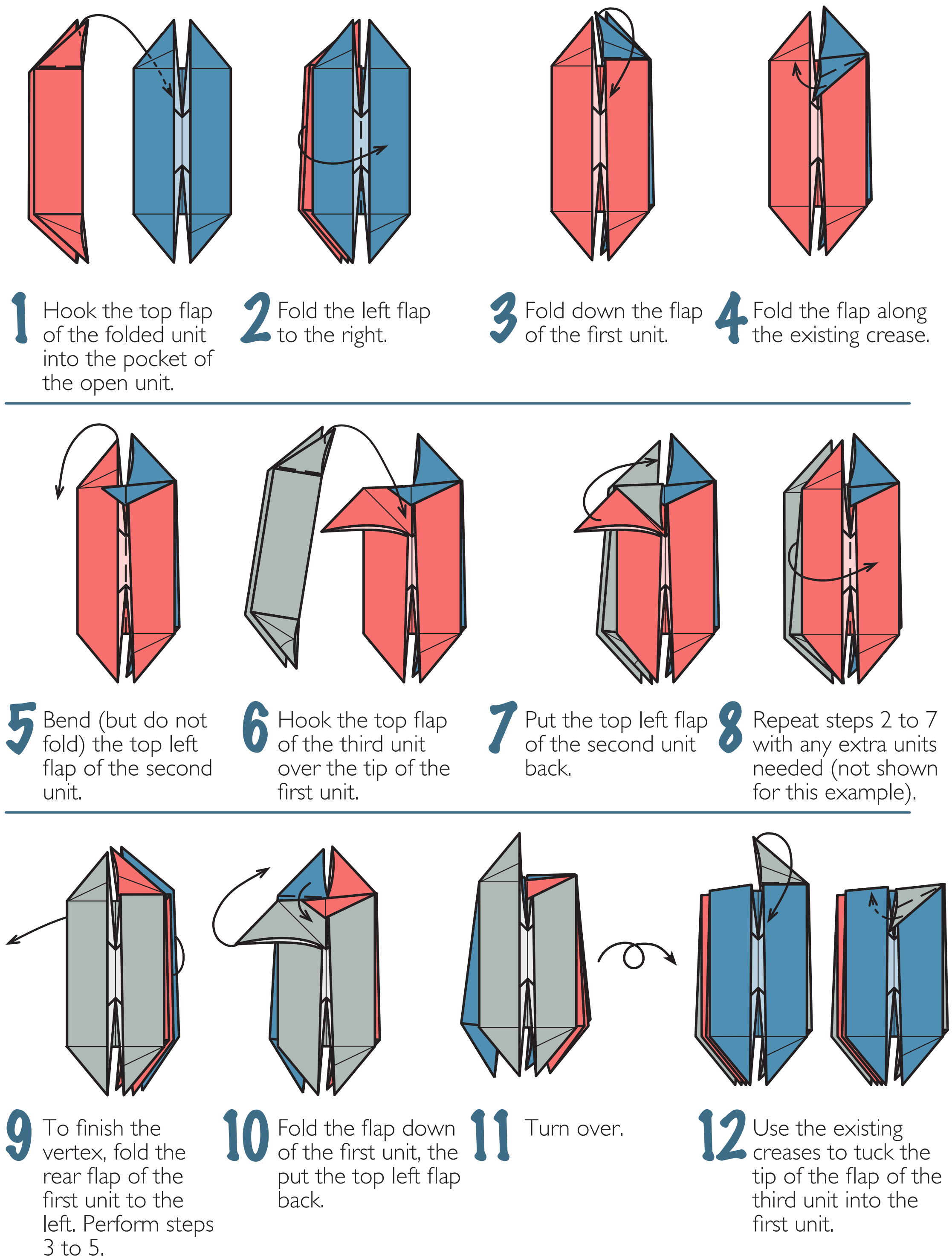

Figure 6: *Assembling the units*

3D Assembly and Colours

To make these flexible assemblies, start with a star of three, four or five colours (Figure 2). Opposite edges of each rhombic face are the same colours. All edges of the same colour are parallel and form a belt that loops around the polyhedron.

Collapsing

Each assembly can collapse in many ways. Figure 7 shows some notable configurations. What other shapes can you find?

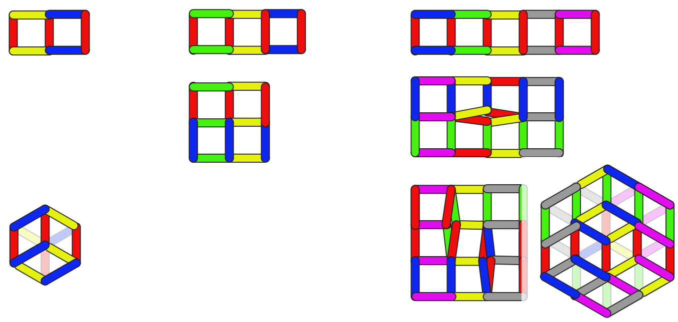

Figure 7: *Collapsing the cube (left), rhombic dodecahedron (middle) and rhombic triacontahedron (right) can make rectangular or hexagonal arrays of squares or triangles.*

Other 3D Assemblies

20 units make a partial icosahedron (Figure 8), a generalisation of the rhombohedron. Pulling the poles apart makes a line and pushing them together makes a pentagonal antiprism (Figure 9). Use longer rectangles for a more effective assembly. In general, an assembly of degree *n* needs 4*n* units.

24 units make a cuboctahedron but this not a good candidate for the Jitterbug transformation. The two causes are the mutual interference of the projecting flaps of paper and the constrained movement at the joints (not enough excess paper).

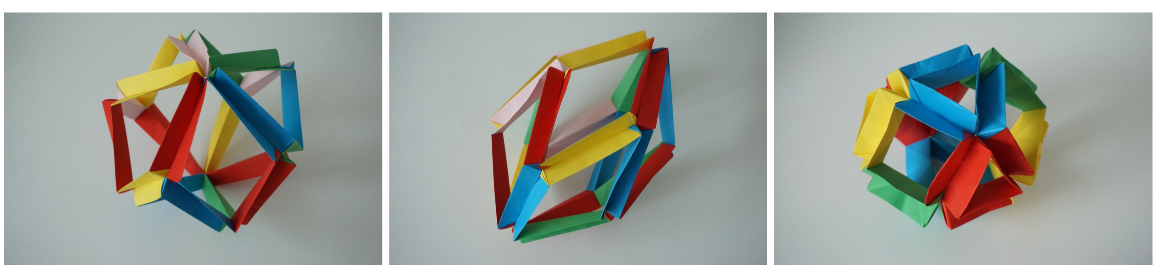

Figure 8: *The partial icosahedron (left and middle) made from 20 units will flex and change shape. However, the cuboctahedron (right) does not flex very well and is not a good candidate for the Jitterbug transformation.*

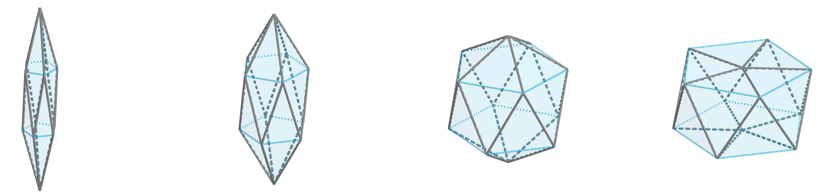

Figure 9: *The partial icosahedron transforming into a pentagonal antiprism*

2D Assembly

Curiously, 2D structures can be harder to assemble than the 3D structures (Figure 10). They also do not flex as well. Reasons include the need for units to to be fully folded in half and the fact that two units do not join well together.

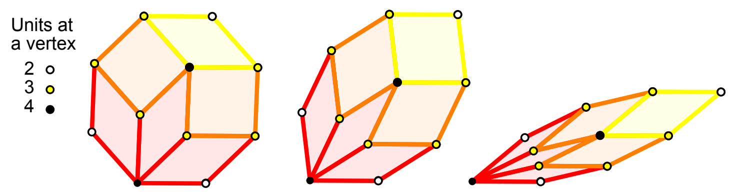

Figure 10: *A 2D assembly of 16 units. The octagon is a zonogon that can collapse into a line.*

Acknowledgements

Thanks to Jorge Pardo for his original *Flexiball* and George Hart for his research into polar zonohedra.

References

[1] S. Biddle and M. Biddle. *The New Origami*. Ebury Press, 1993.

[2] G. Hart. "The Joy of Polar Zonohedra." *Proceedings of Bridges 2021: Mathematics, Art, Music, Architecture, Culture*. D. Swart, F. Farris, and E. Torrence, Eds. Tessellations Publishing, 2021. pp. 7–14. ISSN: 1099-6702. http://archive.bridgesmathart.org/2021/bridges2021-7.html.

[3] T. K. Lam. "Action Modular Origami." *Proceedings of Bridges 2018: Mathematics, Art, Music, Architecture, Education, Culture*. E. Torrence, B. Torrence, C. Séquin, and K. Fenyvesi, Eds. Phoenix, Arizona: Tessellations Publishing, 2018. pp. 563–566. http://archive.bridgesmathart.org/2018/bridges2018-563.html.

[4] T. K. Lam. *Action Modular Origami to intrigue and delight: How to fold paper into geometrical sculptures that move, spin and change shape*. Tarquin, 2018.

[5] T. K. Lam. *Modular Origami: 18 Colorful and Customizable Folded Paper Sculptures*. Schiffer Craft, 2023.

[6] R. J. Lang. Polypolyhedra. https://langorigami.com/article/polypolyhedra/.

[7] B. Loper. *Mind-Blowing Modular Origami: The Art of Polyhedral Paper Folding. Use Origami Math to fold Complex, Innovative Geometric Origami Models*. Tuttle Publishing, 2016.

[8] B. Martín. Flexiball de Jorge Pardo. https://bermarez.com/papiro/54_flexiball_de_jorge_pardo.html.

[9] J. Pardo. "Flexiball." *British Origami*, no. 235, 2005.